Instanton and Superconductivity in Supersymmetric $CP^{N-1}$ Model

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memory of Larkin

- Topological excitations (Pokrovskii talk)
  - instanton discussion in the first visit (1979)

- "hic sunt leones"
  (Theory of fluctuation in superconductor by Varlamov-Larkin, P. 465 Russian edition)
  - discussions of magneto-resistance in high $T_c$, $\tau_\varphi$ (1988)
Fluctuation of superconductor;

Ginzburg-Landau Hamiltonian (N=1)

\[ H_{GL} = \frac{1}{2} |\partial_{\mu} \phi_i + ieA_{\mu} \phi_i|^2 + g|\phi_i|^4 + \frac{1}{4} F_{\mu\nu}^2 \]

(i=1,...,N) renormalization group does not give a stable fixed point at \( d = 4 - \epsilon \) except for large \( N \) (first order transition).

For \( d=3 \), another Hamiltonian, \( CP^{N-1} \) model,

\[ H_{CP^{N-1}} = \frac{1}{2t} (\partial_{\mu} \bar{Z}_i \partial_{\mu} Z_i + (\bar{Z}_i \partial_{\mu} Z_i)^2) \]

(\( \bar{Z}_i Z_i = 1 \))

(renormalization group works for \( 2 < d \))
If there is a fixed point, then the bare couplings become infinite, $g \rightarrow \infty$, $e \rightarrow \infty$,

$$H_{GL} \rightarrow H_{CP^{N-1}}$$

$$\beta(t) = \epsilon t - N t^2 + \cdots$$

($\epsilon = d - 2$) At $d=3$, a nontrivial critical exponent is obtained for $N=1$ (second order transition for a superconductor). (For $N=2$, it is same as classical Heisenberg model)
This gauge invariant $CP^{N-1}$ nonlinear $\sigma$ model is a proto-type of the Anderson localization problem. (1979)

Grassmanian manifold;

\[
\frac{U(N)}{U(N-1) \times U(1)} \rightarrow \frac{U(N)}{U(N-p) \times U(p)}
\]

\[
CP^{N-1} = \frac{U(N)}{U(N-1) \times U(1)}
\]
Vaks-Larkin four-Fermi interaction model

mass generation; mass = superconductor order parameter

gap equation

Renormalization group $\beta$ function is asymptotic free at $d=2$. There is no long range order in $d=2$ (Mermin-Wagner theorem). The average of the generated mass (superconductor gap $\Delta$) has to be vanishing; due to kink excitation.
supersymmetric generalization of these two models

\[ CP^{N-1} \rightarrow \text{supersymmetric } CP^{N-1} \text{ model} \]

\[
S = \int d^2 \sigma d^2 \theta \sqrt{g} g^{\mu \nu} G_{\alpha \beta}(X) D_\mu X^\alpha D_\nu X^\beta
\]

\[
D = \partial_\theta + \theta \partial_z, \quad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}}
\]

\[
\beta_{ij} = \epsilon t - R_{ij} t^2 + \frac{1}{2} \zeta(3) (a_3 - a_6) t^5 + \cdots
\]
(1) The two dimensional supersymmetric CP(1) model

(2) Boundary of the strong coupling and the weak coupling regions: a curve of marginal stability (CMS)

(3) Pseudo gap phase in the high $T_c$ superconductor

(4) Charged deformed instanton

(5) Vortex core state

(6) Equivalent one dimensional spin model
The supersymmetric O(3)/O(2) (=CP$^1$) non-linear $\sigma$ model

\[ \mathcal{L} = \frac{1}{g} \int d^2x \left[ \frac{1}{2} (\partial_\mu n)^2 + \frac{1}{2} \bar{\psi} i \gamma_\mu \partial_\mu \psi + \frac{1}{8} (\bar{\psi}^a \psi^a)^2 \right] \] (1)

(a=1,2,3)

\[ n^2 = 1, \quad n \cdot \psi = 0 \] (2)

Fermionic part: Vaks-Larkin, Gross-Neveu model

$\beta$-function shows asymptotic free behavior ($\beta > 0$) in one loop order, and no higher order corrections generating mass $\sim$ the gap of BCS theory; $\Delta = e^{-c/g}$

There is no long range order in d=2
supersymmetric $CP^{N-1}$ model

$$\mathcal{L} = \int d^4 \theta K(\Phi, \Phi^\dagger)$$

$$= G_{ij} (\partial^\mu \phi^j \partial_\mu \phi^i + i \bar{\psi}^j \gamma^\mu D_\mu \psi^i) - \frac{1}{2} R_{ijkl} \bar{\psi}^j \psi^i \bar{\psi}^l \psi^k$$

(3)

$$\mathcal{L}(CP^1) = G (\partial_\mu \phi^\dagger \partial^\mu \phi - |m|^2 \phi^\dagger \phi + i \bar{\psi} \gamma^\mu \partial_\mu \psi$$

$$- \frac{1 - \phi^\dagger \phi}{1 + \phi^\dagger \phi} \bar{\psi} M \psi$$

$$- \frac{2i}{1 + \phi^\dagger \phi} \phi^\dagger \partial_\mu \phi \bar{\psi} \gamma^\mu \psi + \frac{1}{(1 + \phi^\dagger \phi)^2} (\bar{\psi} \psi)^2)$$

(4)

with $M = m(1 + \gamma_5)/2 + \bar{m}(1 - \gamma_5)/2$, $\psi^T = (\psi_R, \psi_L)$. $G = 2/g(1 + \phi^\dagger \phi)^2$
or equivalently,

\[
\mathcal{L}(CP^1) = G(\partial_{\mu} \phi^\dag \partial^{\mu} \phi - |m|^2 \phi^\dag \phi \\
+ \frac{i}{2}(\psi_L^\dagger \partial_R \psi_L + \psi_R^\dagger \partial_L \psi_R) \\
- \frac{i}{2}(1 + \phi^\dag \phi) \left( m \psi_L^\dagger \psi_R + \bar{m} \psi_R^\dagger \psi_L \right) \\
- \frac{2}{(1 + \phi^\dag \phi)^2} \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \\
- \frac{i}{1 + \phi^\dag \phi} \left[ \psi_L^\dagger \psi_L (\phi^\dag \bar{\partial}_R \phi) + \psi_R^\dagger \psi_R (\phi^\dag \bar{\partial}_L \phi) \right] \right) \tag{5}
\]

with \(\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}\), \(\partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}\).
Twisted mass $m$

Superfield has a mass term $m$ (Higgs mass). The presence of this mass term does not break the $\mathcal{N} = 2$ supersymmetry. This mass term gives the easy axis anisotropy (Ising like).

bosonic part

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{n} \partial^\mu \vec{n} - \frac{1}{2} m^2 (\xi - n_3^2)$$  \hspace{1cm} (6)

with $\xi = 1/g$.

Time dependent dyon solution

$$\phi_{cl} = e^{\sqrt{m^2 - \omega x^2 + i\omega t}}$$
\[ \phi = \tan\left(\frac{\varphi}{2}\right)e^{i\alpha} \]

\[ n = (\sin \varphi \cos \alpha, \sin \varphi \sin \alpha, -\cos \varphi) \]

\[ \phi = \frac{n_1 + in_2}{1 - n_3} \]

**Charge Q**

instanton with charge Q: (\(\phi\) is time dependent)

\[ Q = i\xi \int d^2x \frac{\bar{\phi} \dot{\phi} - \phi \dot{\bar{\phi}}}{(1 + \bar{\phi}\phi)^2} \quad (7) \]

instanton number \(n\):

\[ n = -\frac{i}{4\pi} \int d^2x \epsilon_{ij} \frac{\partial_i \bar{\phi} \partial_j \phi - \partial_j \bar{\phi} \partial_i \phi}{(1 + \bar{\phi}\phi)^2} \quad (8) \]
tension $T$ in BPS form

\[ T = \xi \int d^2x \left[ \left| \frac{\partial_t \phi \pm im\phi}{1 + \phi \phi} \right|^2 + \frac{1}{2} \left| \frac{\partial_i \phi \pm i\epsilon_{ij} \partial_j \phi}{1 + \phi \phi} \right|^2 \right] + 2\pi\xi n + mQ \quad (9) \]

(The first two terms have to vanish.)

\[ \phi(z, t) = \left( \sum_{i=1}^{n} \frac{c_i}{z - z_i} \right) e^{imt} \quad (10) \]
In the four dimensional SQCD, the massive excitations of monopole, dyon, and Noether charges are evaluated by the elliptic integrals. (Seiberg-Witten)

In the two dimensional $CP(1)$ case, the electric charge $Q$ and topological charge $T$, are combined with the (twisted) mass $m$ and the dual mass $m_D$ to form the central charge $Z_{Q,T}$

$$Z_{Q,T} = mQ + m_DT \quad (11)$$

$(Q,T)$ is pseudo particle (charged deformed instanton). $Q,T$ are integers
Figure of the magnetic flux and the charge

— bamboo structure —
The effective twisted superpotential $\tilde{W}$: the twisted chiral superfield $\Sigma$: $\partial\tilde{W}/\partial\Sigma = 0$

\[ \prod_{i=1}^{N} (\sigma + m_i) - \tilde{\Lambda}^N = \prod_{i=1}^{N} (\sigma - e_i) = 0. \quad (12) \]

\[ Z_{12} = 2[\tilde{W}(e_1) - \tilde{W}(e_2)] = \frac{1}{2\pi} [N(e_1 - e_2) - \sum_{i=1}^{2} m_i \ln\left(\frac{e_1 + m_i}{e_2 + m_i}\right)]. \quad (13) \]

$m_1 = -m_2 = -m/2$, $\sigma^2 - \frac{m^2}{4} = \tilde{\Lambda}^2$. $m_D$ (dual mass)

\[ m_D = \frac{i}{\pi} \left[ \sqrt{m^2 + 4\tilde{\Lambda}^2} + \frac{m}{2} \ln\left(\frac{m - \sqrt{m^2 + 4\tilde{\Lambda}^2}}{m + \sqrt{m^2 + 4\tilde{\Lambda}^2}}\right) \right] \quad (14) \]
\[ \tau = -\frac{i}{g} + \frac{\theta}{2\pi}, \]  
\[ \tilde{\Lambda}, \text{ modified by the } \theta \text{ value,} \]  
\[ \frac{\tilde{\Lambda}}{m} = \frac{1}{2} e^{-1 + i\pi \tau} = \frac{1}{2} e^{-1 + \frac{\pi}{g} + \frac{i}{2} \theta}. \]  
A critical point at \( g = \theta = \pi \), \( 4(\tilde{\Lambda}/m)^2 = -1 \). The twisted chiral superfield:

\[ \Sigma = \sigma + \sqrt{2} \psi^\alpha \tilde{\chi}_\alpha + \psi^\alpha \vartheta^\alpha S. \]
Curve of Marginal Stability (CMS)

In the strong coupling region $|m| \ll \Lambda$, BPS states becomes only two states $(Q = 0, T = 1)$ and $(Q = 1, T = 1)$ ($m \approx 0$). In the weak coupling region $|m| \gg \tilde{\Lambda}$, these two states are bounded, and other $(Q, T)$ BPS states appear, and whole instanton contributions appear.

There is a boundary, called as the curve of the marginal stability (CMS) in the complex mass parameter $m^2$, where the restructuring of the BPS states occurs. The curve of marginal stability is
expressed as the following equation (Dorey, Shifman)

$$\text{Re}[\ln \frac{1 + \sqrt{1 + 4\tilde{\Lambda}^2/m^2}}{1 - \sqrt{1 + 4\tilde{\Lambda}^2/m^2}} - 2 \sqrt{1 + 4\tilde{\Lambda}^2/m^2}] = 0.$$  

(18)

a singular point $4\frac{\tilde{\Lambda}^2}{m^2} = -1$, the value of $m_D$ in (14) becomes vanishing. This critical point is realized for $g = \theta = \pi$. 
Curve of marginal stability (CMS)

\((4\Lambda^2 \rightarrow 1)\)
phase diagram of high $T_c$ superconductor

\[ \begin{array}{c}
T \\
\hline
C \\
P \\
S \\
\hline
\delta
\end{array} \]
Pseudo gap phase

PGP = Soliton (monopole) + Dyon (q,T) (weak coupling phase)

Coulomb Phase = Soliton, Dyon [(0,1), (1,1)] \( m \to 0 \) (strong coupling phase)

Ising like easy axis anisotropy

Correlation function is described by Painleve III, same as 2D Ising model for the spin-spin correlation function (T.T. Wu, Cecotti-Vafa)

\[ \Lambda \over m \gg 1 \]
Weak coupling phase and strong coupling phase is separated by a curve of marginal stability (CMS).

In two dimension, there is no long-range order similar to the disorder phase of 2D Ising model.

Kink walls give strip structures or bamboo structure.
Vortex core excitation

Inside the vortex, the excitation of the pseudo-gap phase can be seen in the underdoped region.

Charged deformed instanton (dyon, (Q,T))

recent STM observations of the antisymmetrical charging
Figure of the vortex structure

—- Cross cap (Satsuma) structure —-

vortex core
cross cap
• : charge
STM, Nishida et al (2007)
Equivalent one dimensional spin model

The Hamiltonian $\mathcal{H}$ of n-vector model is

$$\mathcal{H} = -J \sum_{i=1}^{M-1} \vec{S}_i \cdot \vec{S}_{i+1},$$

(19)

with a condition,

$$|\vec{S}_i|^2 = \sum_{m=1}^{n} S_i^2(m) = n.$$  

(20)

$$Z = \left[\left(\frac{nJ}{2kT}\right)^{1-n/2} \Gamma\left(\frac{n}{2}\right) I_{\frac{n}{2}-1}\left(\frac{nJ}{kT}\right)\right]^{M-1}$$

(21)

$I_\nu(z)$ : modified Bessel function. $\nu = \frac{n}{2}$ and $Y = \frac{2J}{kT}$
\[ I_{\nu}(\nu Y) = \frac{1}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \left(\frac{\nu Y}{2}\right)^\nu 4^\nu e^{-\nu Y} \int_0^\infty e^{-F(t)} dt, \]  
with  
\[ F(t) = (\nu + \frac{1}{2})t - (\nu - \frac{1}{2})\ln(1 - e^{-t}) - 2\nu Ye^{-t}. \] 
\[
\nu \to \infty \frac{\partial F(t)}{\partial t} = 0 \text{ (saddle point equation) The two saddle points } t_+ \text{ and } t_-
\]
\[ e^{-t_\pm} = \frac{Y - 1 \pm \sqrt{1 + Y^2}}{2Y}. \]
\[ F(t_{\pm}) = \nu \left( 1 - Y \pm \sqrt{1 + Y^2} - \ln \frac{-1 \pm \sqrt{1 + Y^2}}{2Y^2} \right). \tag{25} \]

The difference becomes
\[ F(t_-) - F(t_+) = 2\nu \left( \frac{1}{2} \ln \frac{1 - \sqrt{1 + Y^2}}{1 + \sqrt{1 + Y^2}} + \sqrt{1 + Y^2} \right). \tag{26} \]

This gives the dominant contribution to the large order behavior; instanton contribution (Hikami-Brezin 1979)
In the $\frac{1}{n}$ expansion, we obtain

$$I_{\nu}(\nu Y) = \frac{1}{2\pi \nu} \frac{e^{\nu \eta}}{(1 + Y^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right),$$

where we obtain $\eta$, by collecting the coefficients in (22),

$$\eta = \sqrt{1 + Y^2} + \frac{1}{2} \ln \frac{\sqrt{1 + Y^2} - 1}{\sqrt{1 + Y^2} + 1}.$$ 

The parameter $t$ is $(1 + Y^2)^{-1/2}$. It is an asymptotic expansion, which glows like $(k-1)!/[F(t_+) - F(t_-)]^k$.

If we identify $\frac{J}{kT} = \frac{\tilde{\Lambda}}{m}$, $(Y^2 = \frac{4\tilde{\Lambda}^2}{m^2})$, we find the expression for $m_D$. 
This one dimensional n-vector model is solved by the transfer matrix method. The transfer matrix is

\[ T = e^{\frac{nJ}{kT} \vec{S} \cdot \vec{S}'} \]  \hspace{1cm} (29)

The eigenvalues are given by

\[ T_l = CI_n^{\frac{n}{2}-1} + l \left( \frac{nJ}{kT} \right), \]  \hspace{1cm} (30)

where \( l = 0, 1, 2, ..., \) and \( C \) is a constant. The spin-spin correlation function for the distance \( r \) is given by

\[ \langle \vec{S}(0) \cdot \vec{S}(r) \rangle = \left( \frac{T_1}{T_0} \right)^r = \left( \frac{I_\nu(\nu Y)}{I_{\nu-1}(\nu Y)} \right)^r, \]  \hspace{1cm} (31)

with \( \nu = \frac{n}{2} \).
When $Y$ is a real, a correlation length $\xi$ is finite. $I_\nu(\nu Y) < I_\nu(\nu Y)$; is no phase transition at finite temperature.

$$\xi^{-1} = \ln\left(\frac{I_{\nu-1}(\nu Y)}{I_\nu(\nu Y)}\right).$$

(32)

When $Y$ is a pure imaginary number $Y = -i|Y|$, there appears a phase transition with the infinite correlation length by the degeneracy of the eigenvalues of the transfer matrix.

When $Y$ is imaginary, the modified Bessel function is expressed by the Bessel function $J$ as $I_\nu(\nu Y) =$
\( e^{-\nu \pi i/2} J_\nu(\nu |Y|) \). We find in the large \( \nu \) limit, there appears a crossing at \( Y = \pm i \),

\[
J_\nu(\nu |Y|) = J_{\nu-1}(\nu |Y|).
\]  (33)
The point $Y = \pm i$ is the critical point, which corresponds to $4\frac{\bar{\Lambda}^2}{m^2} = -1$. Since the Bessel function $J_\nu(z)$ is an oscillating function, there appear successive degeneracies for $\nu (\nu = \frac{n}{2} - 1 + l)$.

The successive transitions due to the degeneracy of the angular quantum numbers $l = 0, 1, 2, ..., $ which represent s,p,d,f,....,states.

Such successive transitions give a cut in the large $n$ limit beyond $|Y| > 1$, which is a low temperature phase. The mass of the inverse of the correlation length $\xi$, which is finite in the high temperature region, becomes zero below the transition temperature $T_c$ for $\theta = \pi$. 
Note: such phase transitions also appear for the one dimensional n-vector model, with a real positive $\frac{J}{kt}$, and for $n < 1$, (Balian and Toulouse).
Ising-like correlation function (Painlevé III equation)

P II \rightarrow \text{Airy, Gross-Witten model}; \ u'' = 2u^3 + xu

P III \rightarrow \text{chiral random matrix model (Bessel kernel)};
\[ u'' = \frac{1}{u}(u')^2 - \frac{1}{x}u' + \frac{1}{x}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u} \]

Cecotti-Vafa \( tt^* \) equation = Painlevé III equation = T.T. Wu Ising solution; quantum cohomology
\[ \partial_z \partial_{\bar{z}} q_i + e^{q_{i+1} - q_i} - e^{q_i - q_{i-1}} = 0 \]
\[ < \bar{x} | x >^{-1} = \sum_{n=0}^{\infty} |\beta|^{2n} P_n \]
$P_n$: polynomial of $(-\log|\beta| - 2\gamma)$ with $2n+1$ degree.
Ising model $t = \frac{R}{\xi}, R = \sqrt{M^2 + N^2} \rightarrow PIII.$
Random matrix model (Painlevé III)

\[ M = \begin{pmatrix} 0 & C^\dagger \\ C & 0 \end{pmatrix} \]

Bessel kernel:

\[
K(z_1, z_2) = \frac{N}{2\pi} \int_{-\infty}^{\infty} dt \int \frac{du}{2\pi i} \frac{1}{u - it} e^{i\frac{1}{u} - iN^2 tz_1 + N^2 uz_2 + N(z_1 - z_2)}
\]

Fredholm determinant; probability of level spacing; \( H(q,p) \);

\[
\psi'' + \frac{1}{x} \psi' = \frac{1}{2} \sin(2\psi)
\]

\[
q(s) = \cos(\psi(s)), x^2 = s
\]

\((\psi \to i\psi) \to\) Ising case
Reduction to $O(n)$ vector model

supersymmetric $CP^1 \rightarrow O(n)$ vector model ($n \rightarrow \infty$)

$AdS_2$ (2d Liouville theory) $\rightarrow$ 2d supersymmetric QCD : (gauge/string correspondence)

Polyakov (conjecture for higher spin operators)

$AdS_{d+1} \rightarrow d$-dimensional $O(n)$ vector model ($n \rightarrow \infty$)
Conclusion

Supersymmetric $CP^1$ model in $d=2$ with a twisted mass can be solved, and can be applied to the superconducting fluctuation phase.

Interesting topological excitations, instantons with a charge, solitons and dyons are obtained. (Exact CMS )

The correlation function is described by 2d Ising model.

Applications may be for high temperature superconductor in the pseudo gap phase, or quantum Hall effect with chiral symmetry ($\mathcal{N} = 2$ supersymmetry).