Branch-cut singularities in the thermodynamics of Fermi liquid systems

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an experiment (Technion group, 2003) has found a puzzling temperature behavior of the spin susceptibility (at T>2K) which contradicts the standard Fermi-liquid phenomenology

motivated by this experiment, we develop a theory which extends the existing microscopic theory of the Fermi-liquid systems by inclusion of branch-cut singularities

the extended theory resolves the puzzle of the observed behavior (including the sign) of the spin susceptibility in Si-MOSFETs

the work shows that the theory of the Fermi liquid systems is incomplete unless the branch-cut singularities are included

non Fermi liquid corrections within the Fermi liquid state
Motivation of the work:
Measurements of the spin susceptibility in silicon MOSFETs

SdH oscillations clearly indicate the presence of the Fermi surface: i.e. according Landau (1956) this is a Fermi Liquid system.
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Prus et.al. (2003)

\[ n\mu_B^2 / \chi k_B(K) \]

Pauli

\[ \epsilon_F \simeq 30K \div 40K \]

\[ (T/\epsilon_F)^2 \]

will be invisible!

\[ \delta\chi / \chi = - (T/\epsilon_F)^2 \]

Thick line is Fermi gas:

\[ \delta\chi / \chi = - (T/\epsilon_F)^2 \]
The principle of the measurement:

\[ \delta \mu = e \delta \varphi \]

\[-\frac{\delta \mu}{\delta B} = \frac{1}{C} \frac{\delta n}{\delta B} \]

measured in the experiment

\[ F(n, B) \text{ - free energy} \quad \mu = \frac{\partial F}{\partial n}, \quad M = -\frac{\partial F}{\partial B} \quad \Rightarrow \quad \frac{\partial \mu}{\partial B} = -\frac{\partial M}{\partial n} \]

\[ M(B, T, n) = \text{const} + \int_{n_0}^{n} dn \frac{\partial M}{\partial n} \]
Search for non analyticity:

\[ \int d\omega \coth(\beta \omega / 2) f(\omega) \]

If \( f \) is smooth and regular in the vicinity of \( f=0 \), the standard Sommerfeld expansion will involve even powers of \( T \) only.

\[ (T/\epsilon_F)^2 \]

How to avoid Sommerfeld expansion?
Theory of the Fermi Liquid systems

**Pole singularities**

Landau quasiparticles appear as poles of a single-particle correlation function

\[ Z = \frac{\epsilon - \epsilon_p + \mu + i\delta}{\epsilon - \epsilon_p + \mu + i\delta} \]

**Collective modes**

appear as poles of a two-particle correlation function

\[ \chi \frac{\omega^2_q}{\omega^2 - \omega^2_q + i\delta} \]

Collective modes have small phase space.

what else remains in a Fermi liquid system?
Theory of the Fermi Liquid systems  (extension of the Fermi-liquid theory)

what else remains?

Branch cut (edge) singularities

“Edge modes” appear as the threshold to the particle-hole continuum.

Interaction allows for a rescattering of pairs of the quasiparticles

As a result of the rescattering induced by interactions, the branch-cut singularities generate non-analyticities in the thermodynamic potential which reveal themselves via anomalously strong temperature dependences.

States near the thresholds of p-h continuum are highly sensitive to temperature smearing and therefore are important for temperature dependences.
Three channels of rescattering

particle-hole pairs (zero-sound) $p+q$ $q, \omega \sim 0$ $p$

particle-particle pairs (Cooper) $p+q$ $-p$

$2p_F$ - pairs (Kohn) $p+q$ $-p$

In thermodynamics, the non-analyticities associated with branch-cuts enter via ring diagrams, i.e., ladders which are closed onto themselves.

$$\delta \Omega (\Delta) =$$

For the ladder diagrams, the constraints imposed by the conservation of the momentum and energy are most effective because they are applied to a minimal number of quasiparticles. In this way, the dominant terms are generated in the thermodynamic potential. In ladders the non-analyticities associated with branch-cuts are not smeared out by subsequent integrations.
Particle-hole ring in the thermodynamic potential (harmonics)

Section of rescattering

\[ [GG]_{q, \omega, \Delta} = -\nu + \nu S(\theta)_{q, \omega, \Delta} \]

Dynamic particle-hole section

\[ S(\theta)_{q, \omega, \Delta} = \frac{\omega}{\omega + \Delta - v_F q \cos \theta} \]

\[ \tilde{\omega} = \omega + \Delta \]

\[ \Delta = 2(g \mu_B/2)H \]

In fermi-liquid we use angular harmonics of the interaction amplitudes

\[ \delta \Omega(\Delta) = \Gamma_n + \tilde{A}_m + \ldots \]

Angular harmonics of the dynamic particle-hole propagation function

\[ S_{n-m} = \int (d\theta / 2\pi) S(\theta) e^{i(n-m)\theta} \]

\[ \Gamma(\theta_1 - \theta_2) \rightarrow \text{harmonics} \rightarrow \sum \Gamma_n e^{in(\theta_1 - \theta_2)} \]
Particle-hole (zero-sound) ring in the thermodynamic potential

Section of rescattering

\[ p + q \rightarrow p \]
\[ q, \omega \sim 0 \]

\[ [GG]_{q, \omega, \Delta} = -\nu + \nu S(\theta)_{q, \omega, \Delta} \]

Dynamic particle-hole section

\[ S(\theta)_{q, \omega, \Delta} = \frac{\omega}{\omega + \Delta - \nu_F q \cos(\theta)} \]
\[ \tilde{\omega} = \omega + \Delta \]
\[ \Delta = 2(g\mu_B/2)H \]

\[ S_{n,m} = \int \frac{d\theta}{2\pi} S(\theta)e^{i(n-m)\theta} \]
\[ S_{nn} = S_0 = \frac{\omega}{\sqrt{\tilde{\omega}^2 - q^2}} \]
\[ S_{n\neq m} = \left( \frac{\tilde{\omega} - \sqrt{\tilde{\omega}^2 - q^2}}{q} \right)^{|n-m|} S_0 \]

\[ \delta\Omega(\Delta) = \frac{\gamma}{\Gamma_0} \]
\[ \Gamma_n \]
\[ \tilde{\Delta}_m \]

For a single dominant harmonics, e.g., \[ \gamma = \Gamma_0 \]

\[ \delta\Omega(\Delta) = -\left( \frac{\nu}{2\epsilon_F} \right) \int d\omega \coth \frac{\beta \omega}{2} \frac{qdq}{2\pi} \text{Im} \ln \frac{1}{1 + \gamma S_0(q, \omega, \Delta)} \]
Pole in the **complex-q** plane ("Regge pole")

The anomalous part of the magnetization is

\[ M = -\frac{\partial \Omega}{\partial H} \]

\[ \delta M = \int \frac{d\omega}{2\pi} \coth \frac{\beta \omega}{2} \quad \text{Im} \int_0^\infty \frac{qdq}{\pi} \frac{\omega}{\omega^2 - (q\nu_F)^2} \frac{\gamma \omega}{\gamma \omega + \sqrt{\omega^2 - (q\nu_F)^2}} \]

The q-integration is non-vanishing when the pole in the **complex-q** plane is on the imaginary axis

\[ \gamma \omega + \sqrt{\omega^2 - q^2} = 0 \]

Taking the residue we obtain:

\[ \delta M = -\frac{\nu}{2\epsilon_F} \int_{-\Delta}^{\Delta/1+\gamma} \left( \omega + \Delta \right) \coth \frac{\beta \omega}{2} d\omega \]

\[ \delta \chi_{e-h} = -2\nu \frac{T}{\epsilon_F} \left( \ln \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma} \right) \]

A “cocktail” of pairs of quasi-particles and the collective mode (spin wave; Silin) can be effectively described by a pole in the complex momentum plane (like in the high-energy physics).

Pole enters into the **imaginary** axis of the complex q-plane when

\[ -\Delta < \omega < -\Delta/(1+\gamma) \]

A non-analytic contribution >> than in the regular FL problems with 3rd law of thermodynamics

*opposite sign compared to experiment*
Does the linear-T term contradict the third law of the thermodynamics?

it is indirectly assumed above that the thermodynamic potential has a regular expansion near $H, T = 0$

However, for non-analytic function the paramagnetic behavior can survive to zero temperature $\delta M = THm_\gamma(H/T)$

$$H, T \to 0 \quad \delta M \propto HT \quad T > H$$

$$\delta M \propto H^2 \quad T < H$$

(Thermal expansion coefficient)

“paramagnetic behavior”

$\chi^{-1} \sim T + \text{const}$

Third law:

$S_{T \to 0} = 0$

Maxwell’s relation:

$$\left(\frac{\partial M}{\partial T}\right)_H = \left(\frac{\partial S}{\partial H}\right)_T$$

Misawa (1999) emphasized the non-analyticity of $\delta \Omega(H, T)$ and guessed (incorrectly) its non-analytic form.
Partial list of references (theory)

The early work in 3d: G.M. Eliashberg (logT-corrections in the specific heat).

Works in 2D

G. Y. Chitov and A. J. Millis, Phys. Rev. Lett. 86, 5337 (2001);
A. V. Chubukov and D. L. Maslov, Phys. Rev. B 68, 155113 (2003);
A. V. Chubukov, D. L. Maslov, S. Gangadharaiah, and L. I. Glazman
Phys. Rev. B 71, 205112 (2005);
A. V. Chubukov, D. L. Maslov, S. Gangadharaiah, and L. I. Glazman,

D. L. Maslov, A. V. Chubukov, R. Saha, cond-mat/0609102
Nonanalytic Magnetic Response of Fermi- and non-Fermi Liquids
After the transformations that are usual in the theory of a Fermi fluid, for an arbitrary relation between $k$ and $\omega$, we obtain for the susceptibility the expression
\[ \chi = \int d\epsilon \, dp \, Sp \left\{ \sigma \frac{\partial G^{-1}}{\partial \epsilon} (GG - [GG]_{k=0}) [1 + \Gamma (GG - \{GG\}_{k=0})] \frac{\partial G^{-1}}{\partial \epsilon} \sigma \right\} . \] (21)

Here the integration is carried out only over the region close to the Fermi surface, where one may use expression (19) for the Green function. The contribution from the first term in the square brackets leads to the Pauli susceptibility with an effective mass. The renormalized multiplier $a$ in this term is canceled. In the second term, which is a small correction, it may be neglected. On substituting for the amplitude $\Gamma$ the expression obtained above, we get the static susceptibility
\[ \chi = \chi_0 \left\{ 1 + \frac{1}{2} g_4 \left[ 1 + \left( 1 + g_4 \ln \frac{\epsilon_p}{\max T, \mu H} \right)^{-1} \right] \right\} . \] (22)

In the region of a normal metal, the formula obtained is applicable at as small fields and temperatures as is desired, and it implies a slow decrease of the susceptibility with decrease of temperature. In the superconducting and antiferromagnetic regions, the expression obtained is applicable only for sufficiently high temperatures, where the effective interaction is small. With further decrease of temperature, as is seen from the exact solutions, the drop becomes more rapid.
Calculations (in harmonics) of two-section terms, $\delta \chi(2)$

Calculations (in harmonics) can be done in the zero-sound and Cooper channels separately and demonstrate (surprisingly) that:

in the spin susceptibility the two-section terms in all channels are dominated by the scattering sharply peaked near the backward direction, $\Gamma(\pi)$. 

$$\delta \chi(2) = \nu \frac{T}{\epsilon_F} \left| \sum (-1^n \Gamma_n) \right|^2 ;$$

$$\sum (\Gamma_n e^{in(\theta_1 - \theta_2)}) \rightarrow \sum (-1^n \Gamma_n) = \Gamma(\pi)$$
Backward scattering amplitude $\Gamma(\pi)$ can be read in three different ways

two-section terms in the spin susceptibility are dominated by the scattering sharply peaked near the backward direction: $\theta_1 - \theta_2 \approx \pi$

\[
\begin{align*}
\delta \chi_{(2)}^{zs} &= \nu \frac{T}{\epsilon_F} \left| \sum (-1^n \Gamma_n^{zs}) \right|^2 \\
\delta \chi_{(2)}^{C} &= \nu \frac{T}{\epsilon_F} \left| \sum (-1^n \Gamma_n^{C}) \right|^2
\end{align*}
\]


$p-h$ (zero-sound)  
\[
\begin{array}{c}
p + q \\
\Gamma(\pi) \\
p \\
-p + k
\end{array} \quad \begin{array}{c}
-p + q + k \\
p + q \\
\Gamma(\pi) \\
-p + k
\end{array}
\]

$p-p$ (Cooper)  
\[
\begin{array}{c}
p + q \\
\Gamma(\pi) \\
-p + q + k \\
-p + k
\end{array} \quad \begin{array}{c}
p + q \\
\Gamma(\pi) \\
p \\
-p + q + k
\end{array}
\]

$2p_F$ (Kohn)  
\[
\begin{array}{c}
p \\
\Gamma(\pi) \\
-p + k \\
-p + q + k
\end{array} \quad \begin{array}{c}
p + q \\
\Gamma(\pi) \\
-p + k \\
-p + q + k
\end{array}
\]
Backward scattering was a starting point of many previous works discussing linear in $T$ corrections in $\chi$. This was by analogy with 1D, e.g., BKV.

★ The diagram with **two rescattering sections** dominated by the **backward** scattering can be read in **three different ways**.

★ The diagram in the zero-sound channel can be **twisted** so as to describe the rescattering in the Cooper channel or two sections in the $2p_F$-scattering channel.

Calculation can be done in each channel separately and demonstrates:

\[
\delta \chi^{zs;C}_{(2)} = v \frac{T}{\varepsilon_F} \left| \Gamma(\pi) \right|^2
\]

\[
\Gamma(\pi) = \sum (-1)^n \Gamma^{zs;C}_n
\]

Backward scattering was a starting point of many previous works discussing linear in $T$ corrections in $\chi$. This was by analogy with 1D, e.g., BKV.
Central point: complete overlapping of two-sections terms

two-section term in the spin susceptibility is dominated by the scattering sharply peaked near the backward direction. The diagram with two rescattering sections dominated by the backward scattering can be read in three different ways. This fact leads to far reaching consequences for thermodynamics.

★Problems:   a) double counting
b) near the backward scattering angle the rescattering in the Cooper channel may intervene

Only dynamic sections are important for non-analytic corrections. (Static parts are used for renormalizations of the amplitudes.) So, in the discussion of two dynamic sections many static can be involved.

\[ \Gamma(p, -p + q + k, -p + k, p + q) \]

For \( \Gamma_n^C > 0 \)

\[ \Gamma_n^C(T) = \frac{\Gamma_n^C}{1 + \Gamma_n^C \ln \epsilon_F / T} \approx 1 / \ln(\epsilon_F / T) \]
Resolution of the problems a) and b)

We give the two-section term to the Cooper channel ladder where it gets killed:

\[ \delta \chi' = \delta \chi_{e-h} - \delta \chi_{(2)} \]

We resolve the problem of a) the double counting and b) logarithmic corrections by calculating the term with two rescattering sections within the Cooper channel ladder where the logarithmic renormalizations originate.

\[ \delta \chi = -2 \nu \frac{T}{\epsilon_F} \left( \frac{\gamma^2}{2} + \ln \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma} \right) \]

resolution of the sign problem
Resolution of the *sign*-problem and avoiding the double counting

The two-section term is sent to and counted in the Cooper channel ladder where it gets killed (!)

There is no overlap for **three-section** term

$$\delta \chi_{(3)} = \left( \frac{T}{\epsilon_F} \right) \nu \int \alpha(\theta_1 \theta_2 \theta_3) \Gamma(\theta_1 - \theta_2) \Gamma(\theta_2 - \theta_3) \Gamma(\theta_3 - \theta_1) d\theta_1 d\theta_2 d\theta_3$$

backscattering is inessential
\( \chi \) decreases with temperature

\[
\gamma = \Gamma_0
\]

\[
\delta \chi = -2\nu \frac{T}{\varepsilon_F} \left( \gamma^2/2 + \ln \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma} \right)
\]

\[
f(\gamma) = \frac{\gamma^2}{2} + \ln \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}
\]

---

Previous works:

\( \chi \) increases with temperature

\[
\begin{align*}
\gamma & = \Gamma_n = \Gamma_{-n} \\
\delta \chi_{n\neq 0} & = -2\nu \left( \frac{T}{\varepsilon_F} \right) [f(\Gamma_n) + f(\Gamma_{-n}) + \phi_n]
\end{align*}
\]

\[
f(\gamma) \sim 0.3 \quad \text{for} \quad \gamma \sim 1 \\
f(\gamma) \sim 0.7 \quad \text{for} \quad \gamma \sim 1.5
\]

\[ \gamma = \Gamma_n = \Gamma_{-n} \]

\[ \delta(n/\chi_{n\neq0}) = T[f(\Gamma_n) + f(\Gamma_{-n}) + \phi_n] \]

\[ f(\gamma = 1) \sim 0.3 \]

\[ \phi_{n=1}(\gamma = 1) \sim 0.24 \]

Few harmonics may be involved
Disorder kills the non-analytic contributions:
a) smears edge singularities
b) harmonics higher than n=0 become ineffective

$\Gamma_n \sim 1$

$g\mu_B/2 = 1$

$\partial(n/\chi)/\partial T = 1$

$r_s = 3 \div 4$

$1/\tau : 2K$
The recently measured spin susceptibility of the two dimensional electron gas exhibits a strong dependence on temperature, which is incompatible with the standard Fermi liquid phenomenology. Here we show that the observed temperature behavior is inherent to ballistic two dimensional electrons.

Besides the single-particle and collective excitations, the thermodynamics of Fermi liquid systems includes effects of the branch-cut singularities originating from the edges of the continuum of pairs of quasiparticles. As a result of the rescattering induced by interactions, the branch-cut singularities generate non-analyticities in the thermodynamic potential which reveal themselves in anomalous temperature dependences.

Calculation of the spin susceptibility in such a situation requires a non-perturbative treatment of the interactions. As in high-energy physics, a “cocktail” of the collective excitations and pairs of quasi-particles can be effectively described by a pole in the complex momentum plane.

This analysis provides a natural explanation for the observed temperature dependence of the spin susceptibility, both in sign and magnitude.

The sign-problem which has been analyzed in this work may have consequences for the physics of the quantum critical point near the ferromagnetic instability.
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