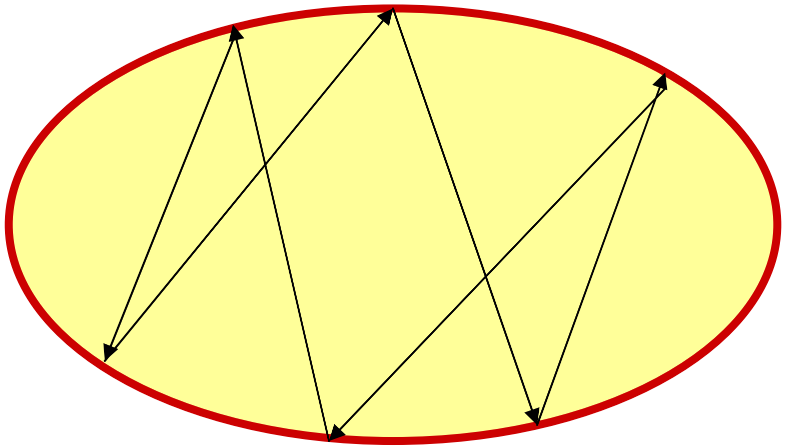
Interference in clean billiards.

K.B. Efetov, V.R. Kogan, and A.I. Larkin



Common knowledge of 2007: energy levels in small metal particles (quantum billiards are described by Wigner-Dyson (WD) statistics (Random Matrix Theory).

Disorder

First use (when it was not common knowledge): disordered grains, Gorkov and Eliashberg (1965).

Explicit derivation using a supermatrix sigma-model (Efetov (1982))

$$\boxed{F_{dif}[Q] = \frac{\pi\nu}{8} Str \int [D(\nabla Q)^2 + i(\omega + i\delta)\Lambda Q] d\mathbf{r}}$$

$$\boxed{Q^2 = 1}$$

Zero-dimensional version:

$$\boxed{F_0[Q] = \frac{\pi\nu i(\omega + i\delta)V}{8} Str(\Lambda Q)}$$

V-volume

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Clean billiards

Conjecture: WD statistics for classically chaotic billiards.
O. Bohigas, M.J. Giannoni, C. Schmit (1984)

Explicit derivation:

1. Attempts to derive supersymmetric ballistic sigma-model (Muzykantskii and Khmelnitskii (1995), Agam, Andreev, Simons, Altshuler (1995)) (not ok).

2. Diagrammatics and phenomenological ballistic sigma-model with a ‘regularizer’, introducing the Ehrenfest time (Aleiner and Larkin (AL), 1997).

3. Applying the AL idea to periodic orbit theory (Sieber and Richter (2001), Muller, Heussler, Braun, Haake and Altland (2004, 2006).

The most probable scenario (Aleiner, Larkin (1996)): a ballistic σ -model with a "regularizer" (semi-phenomenological).

$$F=\frac{\pi\nu}{4}Str\int d\vec{r}d\vec{n}\left[2v_0\Lambda\overline{T}_n(r)\vec{n}\vec{\nabla}T_n(r)+i\frac{\omega+i\delta}{2}\Lambda Q_n(r)+b\left(\frac{\partial Q_n(r)}{\partial\vec{n}}\right)^2\right]$$

$$Q_n=\overline{T}_n\Lambda T_n$$

$$\overline{T}_nT_n=1$$

$$Q_n^2(r)=1$$

$$\vec{n}^2=1$$

Can one really derive such a model microscopically? How to come to the 0D limit?

After the averaging over the spectrum one has:

$$L=L_0+L_{\text{int}}$$

$$L_0=-i\int\overline{\psi}(r)\left(\hat{H}+\frac{\omega+i\delta}{2}\Lambda\right)\psi(r)dr$$

$$L_{\text{int}}=\frac{W^2}{2}\left[\int\overline{\psi}(r)\psi(r)dr\right]^2$$

$$\langle\ldots\rangle_{\varepsilon_0}=\int\frac{(\ldots)}{\left(2\pi N^2\Delta^2\right)^{1/2}}\exp\left(-\frac{(\varepsilon-\varepsilon_0)^2}{2N^2\Delta^2}\right)d\varepsilon$$

$$W=N\Delta$$

Δ is the mean level spacing

Physical meaning of the regularizer:
mixing of classical trajectories.

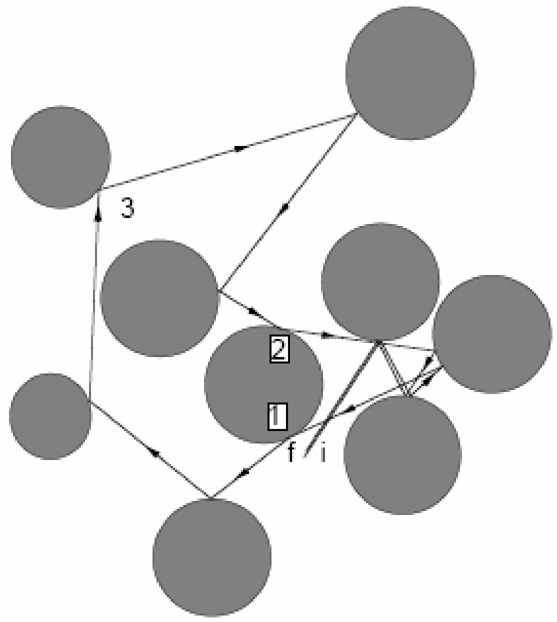


FIG. 2. The classical trajectory corresponding to the probability of return at the initial point with the momentum opposite to the initial one. In the "Lyapunov region" the initial "i-1" and final "2-f" fragments of the trajectory are governed by the same potential.

Aleiner&Larkin (1996)

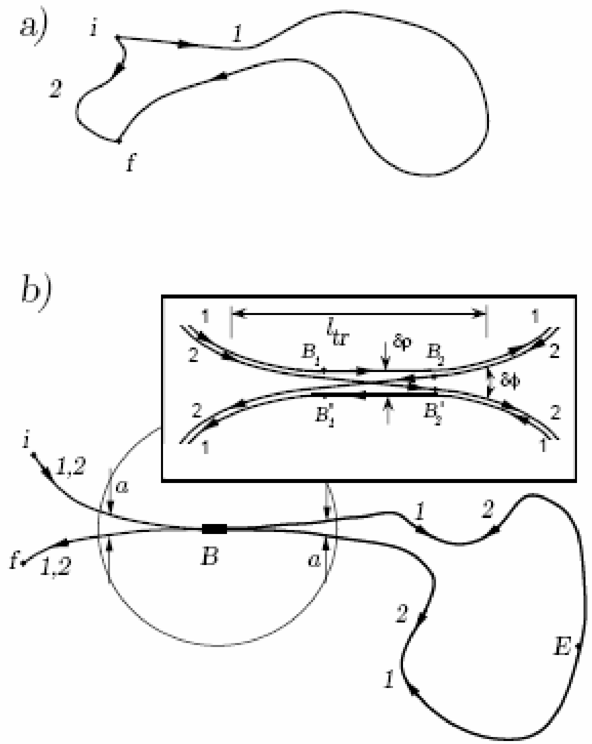
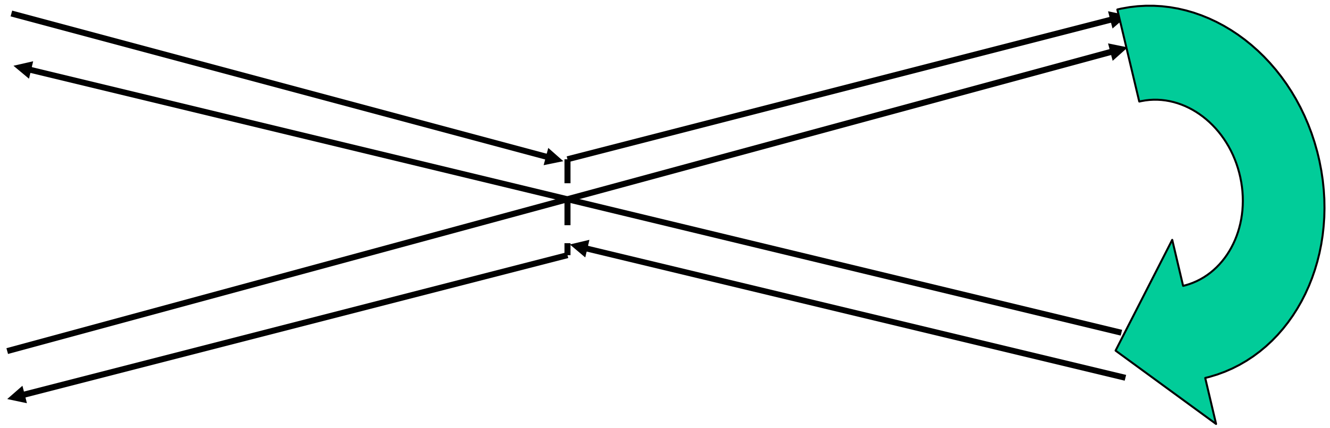


FIG. 3. Examples of the classical (a) non-coherent and (b) coherent paths between points i and f. The scatterers are not shown and the paths are straighten for clarity. Encircled is the Lyapunov region. The region of the quantum switch between trajectories (marked by the rectangular) is blown up on the inset.

Two problems.

1. How to derive a sigma-model in the clean limit? (No saddle point approximation!)

2. How do the trajectories mix without impurities? (Fictitious impurities due to AL describing diffraction but what is it?)



 Impurity line

$$\Delta D(1,2) = \int d3 \frac{C(3,\bar{3})}{\pi \nu \tau_q} \frac{\partial D(1,3)}{\partial \phi_3} \frac{\partial D(\bar{3},2)}{\partial \phi_3}$$

$$j = (\vec{n}, \vec{R})$$
$$\bar{j} = (-\vec{n}, \vec{R})$$

τ_q -scattering time for the fictitious impurities

D-diffuson, C-cooperon, L-Liouville operator

Ehrenfest time $t_E = \lambda^{-1} \ln(\lambda \tau_q)$

λ -is the Lyapunov exponent

What about writing anything similar without introducing fictitious impurities?

Q. Is this the natural spreading of the wave packet?

A. No, although the uncertainty principle is important, the spreading happens at a curved surface.

1. Superbosonization: exact supermatrix ‘sigma-model’.

The Hamiltonian *H*

$$Z[J] = \int \exp(-\mathcal{L})\mathcal{D}(\psi,\bar{\psi}), \qquad (2)$$

$$\mathcal{L} = -i\int \bar{\psi}(\mathbf{r})\mathcal{H}_J(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}d\mathbf{r}', \qquad (3)$$

$$\mathcal{H}_J(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\left(\hat{H}-\varepsilon+\frac{\omega+i\delta}{2}\Lambda\right) - J(\mathbf{r},\mathbf{r}'), \qquad (4)$$

$\psi(r)$ are 8 -component supervectors

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} - \varepsilon_{\mathrm{F}} + U(\mathbf{r}),$$

L - effective Lagrangian with sources

How to rewrite Z[J] in terms of an integral over supermatrices?

Final representation (exact, no averaging over U(r) has been performed)

$$Z[J] = \int \exp(-F[Q])\mathcal{D}Q,$$
$$F[Q] = \frac{i}{2}\int \mathrm{Str}\,[\mathcal{H}_J(\mathbf{x}) * Q(\mathbf{x}) - i\ln Q(\mathbf{x})]\,d\mathbf{x}.$$

x=(r,n)

* -Moyal product

$$A(\mathbf{x}) * B(\mathbf{x}) = A(\mathbf{x})e^{\frac{i}{2}\left(\overleftarrow{\nabla}_{\mathbf{r}}\overrightarrow{\nabla}_{\mathbf{p}} - \overrightarrow{\nabla}_{\mathbf{r}}\overleftarrow{\nabla}_{\mathbf{p}}\right)}B(\mathbf{x})$$

K.E., G. Schwiete,
K. Takahashi (2004)

The replacement of the integration over $\psi(r)$ by the integration over Q(**x**) is analogous to a reformulation of quantum mechanics in terms of the density matrix.

Where is the ballistic sigma-model?

Saddle point of the action at J=0: $q_0(\mathbf{x}) = i\mathcal{H}^{-1}_{J=0} = g(\mathbf{x})$

where g(**x**) the one particle Green function (without sources)

$\Downarrow$

Using the representation $Q(\mathbf{x}) = T(\mathbf{x}) * \tilde{q}(\mathbf{x}) * \bar{T}(\mathbf{x})$, assuming that T(x) is smooth but q is close to g(**x**) we come to the integral:

$$\nu \int g(\mathbf{x}) d\xi \simeq i\nu \int \frac{d\xi}{\xi(\mathbf{p}) + i\delta\Lambda} = ib + \pi\nu\Lambda$$

$$\xi(\mathbf{p}) = \mathbf{p}^2/2m - \varepsilon_{\mathbf{F}}$$

$\Downarrow$

The modulus of the momentum is pinned to the Fermi surface: $\mathbf{p} = \mathbf{n}p_{\mathbf{F}}, \mathbf{n}^2 = 1$

Warning: In clean billiards levels are discrete and the integration over p cannot be performed before an averaging over the spectrum!

This gives a ballistic σ -model valid at all distances exceeding the wave length λ_F

$$F[Q_{\mathbf{n}}] = \frac{\pi\nu}{2}\mathrm{Str}\int d\mathbf{r}d\mathbf{n}\Big[\Lambda\bar{T}_{\mathbf{n}}(\mathbf{r}) \times \left(v_{\mathrm{F}}\mathbf{n}\nabla_{\mathbf{r}} - p_{\mathrm{F}}^{-1}\nabla_{\mathbf{r}}U(\mathbf{r})\nabla_{\mathbf{n}}\right)T_{\mathbf{n}}(\mathbf{r}) + i\left(\frac{(\omega+i\delta)}{2}\Lambda - J_{\mathbf{n}}(\mathbf{r})\right)Q_{\mathbf{n}}(\mathbf{r})\Big],$$

$$Q_{\mathbf{n}} = T_{\mathbf{n}}\Lambda\bar{T}_{\mathbf{n}}.$$

No problem of ‘mode locking’!

But: no regularizer either!

Everything was done for the bulk. The behavior near surface was not treated carefully enough.

The problem is not only about the sigma-model but it is not clear what to do also with diagrammatics!

New approach to the problem:

expansion in the number of reflections (Balian&Bloch (1970))

$$-G_0(r,r') - \oint_B dl' \frac{\partial G_0(\vec{l},\vec{l}')}{\partial \vec{S}_{l'}} \mu(\vec{l}',r') = m\mu(\vec{l},\vec{r}')$$

$$G(r,r') = G_0(r,r') + \oint_B dl \frac{\partial G_0(\vec{r},\vec{l})}{\partial \vec{S}_l} \mu(\vec{l},r')$$

B-boundary

$$\Downarrow$$

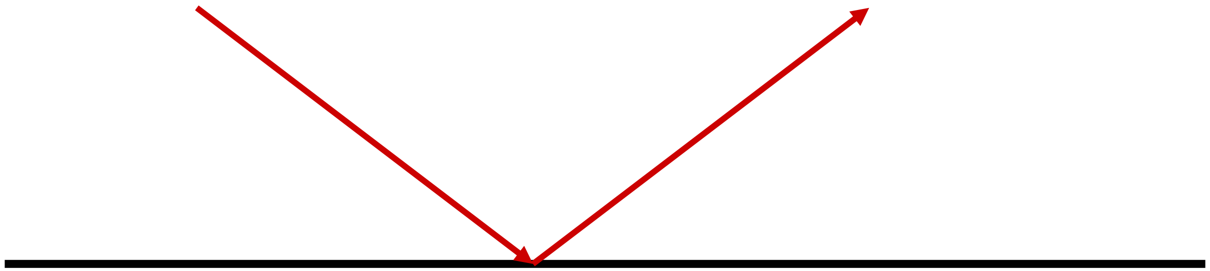
$$G(r,r') = G_0(r,r') - \frac{1}{m}\oint_B dl_1 \frac{\partial G_0(\vec{r},\vec{l}_1)}{\partial \vec{S}_{l_1}} G_0(\vec{l}_1,r')$$
$$+\frac{1}{m^2}\oint_B dl_2 \frac{\partial G_0(\vec{r},\vec{l}_2)}{\partial \vec{S}_{l_2}} \oint_B dl_1 \frac{\partial G_0(\vec{l}_2,\vec{l}_1)}{\partial \vec{S}_{l_1}} G_0(\vec{l}_1,\vec{r}') - \ldots$$

S_l -unit vector normal to the boundary at point l

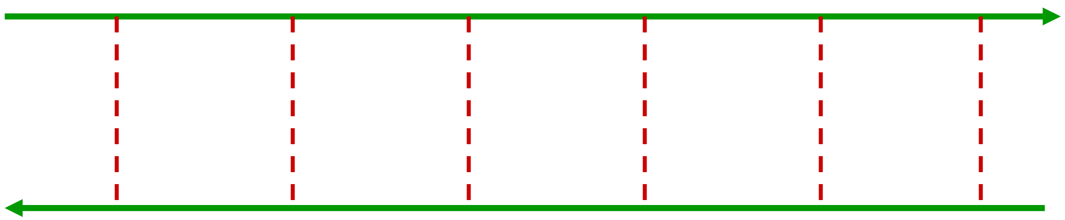
$G_0(r,r')$ -Green function in empty infinite space

Direct analogy to the expansion in an external potential!

Semiclassics: stationary phase approximation.



Diffuson



Averaging over the energy is essential!

Equation for the diffuson.

$$D(\vec{n},\vec{R};\vec{n}',\vec{R}') = D_0(\vec{n},\vec{R};\vec{n}',\vec{R}') - \frac{2\pi\nu}{m^2}\oint dl^r dl^a \int_{\vec{s}^a\vec{n}''\geq 0} d\vec{n}'' \exp\left[-ip_0\left(\vec{n}-\vec{n}''\right)\left(\vec{l}^{\,r}-\vec{l}^{\,a}\right)\right]$$
$$\times\left[\vec{s}^{\,r}\left(p_0\vec{n}+i\nabla/2\right)\right] D_0\left(\vec{n},\vec{R}-\frac{\vec{l}^{\,r}+\vec{l}^{\,a}}{2}\right)\left[\vec{s}^{\,a}\left(p_0\vec{n}''+i\nabla/2\right)\right] D\left(\vec{n}'',\frac{\vec{l}^{\,r}+\vec{l}^{\,a}}{2};\vec{n}',\vec{R}'\right)$$

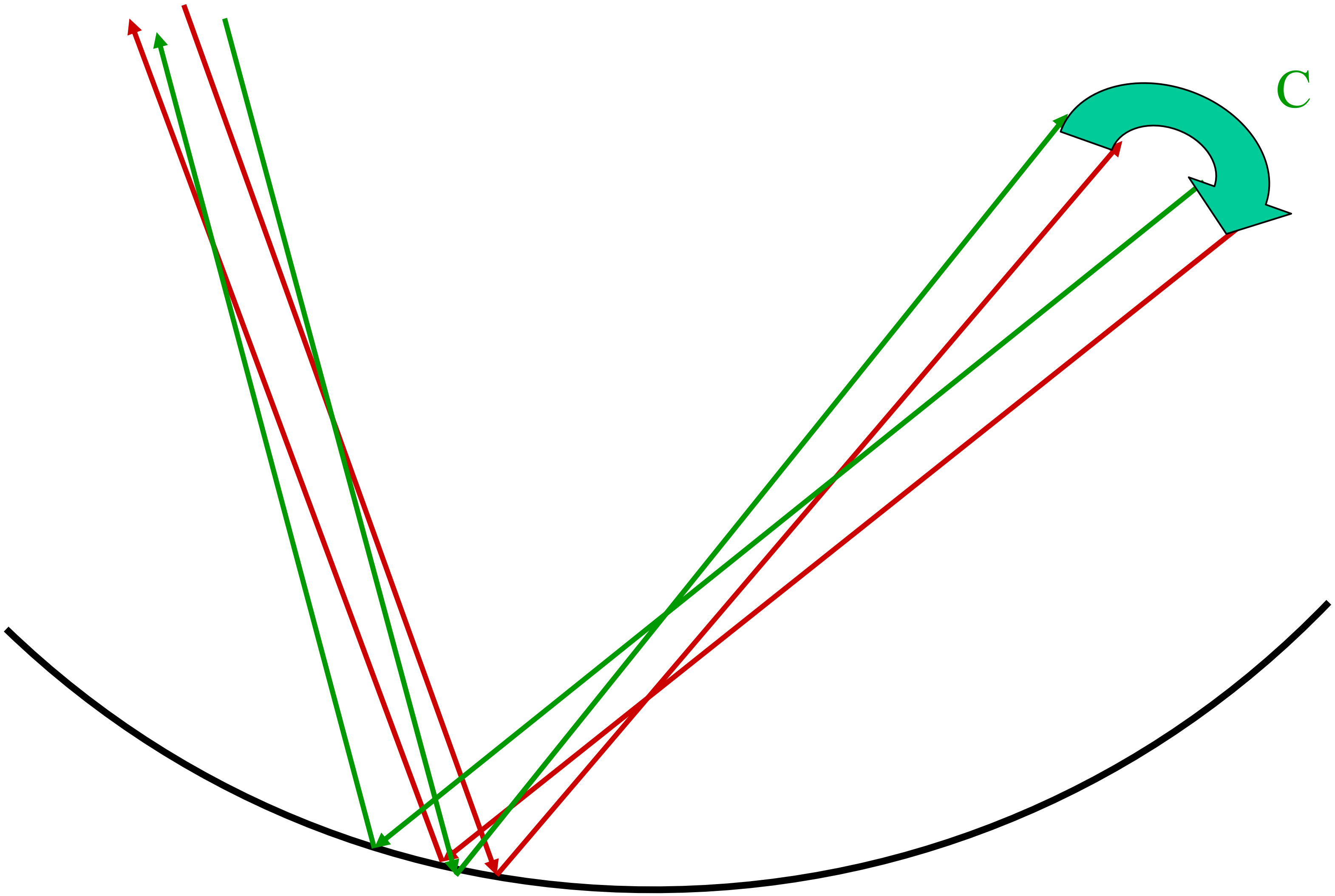
$\vec{s}^{\,r,a}$ is the normal to the boundary at points $l^{r,a}$

$$D_0\left(\vec{n},\vec{R}-\vec{R}'\right) = \int_0^\infty \exp\left(i\left(\omega+i\delta\right)t\right)\delta\left(\vec{R}-\vec{R}'-\frac{p_0}{m}\vec{n}t\right)dt$$

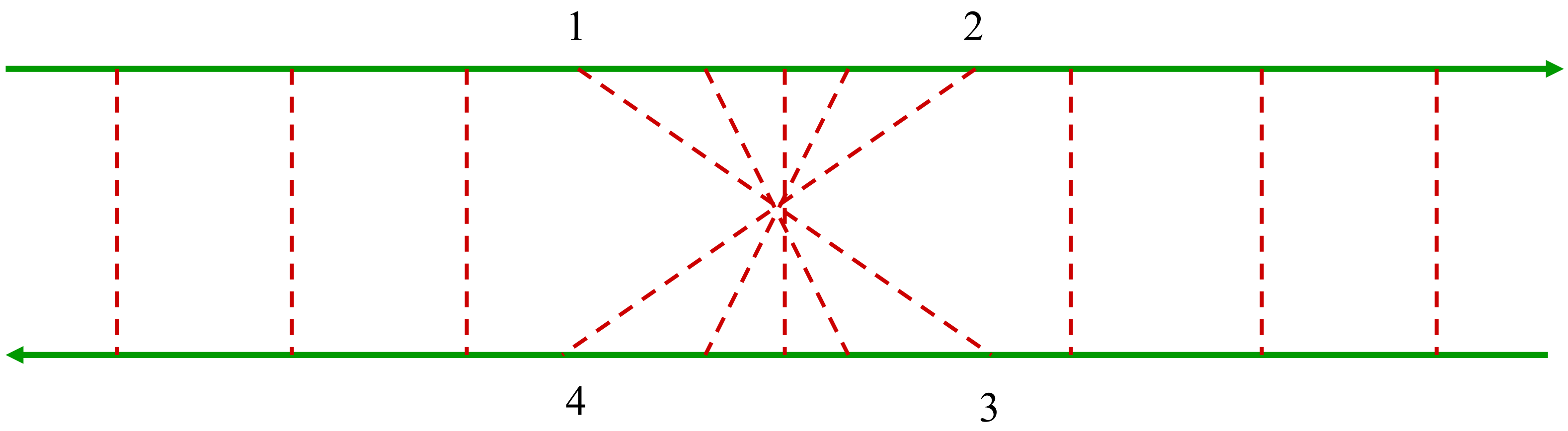
R_c is the curvature, the limit $p_0R_c\to\infty$ corresponds to the flat boundary ⇨ purely classical motion

Quantum corrections: expansion in $\left(p_0R_c\right)^{-1}$

Weak localization correction to the diffuson.



Weak localization correction to the diffuson.



The points 1,2,3,4 are close to each other

Weak localization correction to the diffuson.

$$\delta D\left(\vec{n},\vec{R};\vec{n},\vec{R}\right)=\left(\frac{2\pi\nu}{m^2}\right)^2\oint dL\int d\vec{n}\int p_0\frac{d\delta n_{d\tau}d\delta L}{2\pi}\int p_0\frac{d\delta n_{c\tau}d\delta L'}{2\pi}$$
$$\times\exp\left[ip_0\left(\delta n_{c\tau}\delta L-\delta n_{d\tau}\delta L'\right)\right]\frac{p_0}{|n_s|}\frac{1}{\left(p_0R_c\right)^2}$$
$$\times\frac{\partial D(\vec{n},\vec{R};n+\delta\vec{n}_d/2,\vec{l}(L)+\delta L\vec{\tau}(L)/2)}{\partial\theta}\times\frac{\partial D(-\vec{n}+\delta\vec{n}_d/2,\vec{l}(L)-\delta L\vec{\tau}(L)/2;\vec{n}',\vec{R}')}{\partial\theta}$$
$$\times C\left(\vec{n}+\delta\vec{n}_c/2;\vec{l}(L)+\delta L'\vec{\tau}/2;-\vec{n}+\vec{n}_c/2;\vec{l}(L)-\delta L'\vec{\tau}(L)/2\right)$$

$$\vec{n}=(\cos\theta,\sin\theta),n_s=\cos\theta$$

$(p_0R_c)^{-2}$ -small parameter of the expansion

The regularizer is concentrated on the boundary!

Ehrenfest time: $$\boxed{t_E = \lambda^{-1} \ln\left[(p_0 R_c)^2 \frac{L}{\lambda_F} \right]}$$

L -is the size of the system

The form of the regularizer in the ballistic σ -model (?)

$$\boxed{F_{reg}[Q] = \frac{a}{(p_0 R_c)^2} \oint_L Str\left[\frac{1}{\cos\theta} \frac{\partial Q_n(l)}{\partial\theta} * \frac{\partial Q_n(l)}{\partial\theta} \right] dl,}$$

Conclusions.

1. New diagrammatic technique based on the expansion in the number of collisions is developed. The parameter of expansion is $(p_0 R_c)^{-2}$

2. A weak localization correction has been calculated.

3. The analysis basically confirms the scenario of Aleiner &Larkin, although the regularizer should be pinned to the boundary.